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MAR 77 M MORF, E VERRIEST, J DOBBINS

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ABSTRACT

The Morf-Kailath discrete square-root filtering algorithms are extended to incorporate sensitivity information with respect to the assumed model and statistics. If the model and statistics are incorrect, we presume that another set of model parameters and statistics is available, considered to be the "actual" model. The sensitivity analysis is carried out by computing the so-called "variance ratios" that can be obtained by a second set of orthogonal transforms. These operations can be performed either in covariance or information filter form, as the actual filters. This leads to four basic variants for the propagation of the variance ratios, one of them has been obtained by Bierman.

These ideas can also be applied to other related areas such as smoothing and the dual control problems, where it is of interest to study the sensitivity of the total cost with respect to different models and weights of the cost.

I. Square-Root Filters Applied to Sensitivity Analysis

In this paper we consider the linear discrete-time dynamic model, where the superscript "a" denotes "actual" model parameters

$$x_{i+1} = \phi_i^a x_i + \Gamma_i^a u_i \quad \phi_i^a: n \times n \quad \Gamma_i^a: n \times p \quad (1)$$

$$y_i = H_i^a x_i + v_i \quad H_i^a: m \times n \quad (2)$$

for simplicity, all random variables have zero mean and covariances

$$\text{cov } x_0 = P_0^a \quad (3)$$

$$\text{cov} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} Q_i^a & C_i^a \\ C_i^a & R_i^a \end{bmatrix} \quad (4)$$

The propagation of the square root array is based on the identity in [1]

$$\begin{bmatrix} R_i + H_i P_i H_i' & H_i P_i \phi_i' + C_i \Gamma_i' & C_i \\ \phi_i P_i H_i' + \Gamma_i C_i' & \phi_i P_i \phi_i' + \Gamma_i Q_i \Gamma_i' & \Gamma_i Q_i \\ C_i' & Q_i \Gamma_i' & Q_i \end{bmatrix} = \begin{bmatrix} R_i^c & K_i' & C_i \\ K_i & P_{i+1} + K_i R_i^{-c} K_i' & \Gamma_i Q_i \\ C_i' & Q_i \Gamma_i' & Q_i \end{bmatrix} = M_{3 \times 3} \quad (5)$$

The following factorizations into triangular components are made:

$$M_{3 \times 3} = C_1 C_1' = C_2 C_2' \quad (6)$$

where C_1 is an upper triangular square-root of the LHS of (5) and C_2 a lower triangular square-root of the RHS matrix.

A geometrical interpretation accompanies this representation (cf. [1]). Indeed, letting

$$\tilde{x}_i^a \triangleq [\tilde{y}_i' | \tilde{x}_{i+1}' | \tilde{u}_i' | \tilde{u}_{i-1}']$$

then:

$$C_1^{-1} \tilde{x} \text{ and } C_2^{-1} \tilde{x}$$

are two orthonormal sets (ξ_1 and ξ_2 respectively) for the $(m+n+p)$ -dimensional space spanned by \tilde{x} . In other words, geometrically an orthonormalization process (Gram-Schmidt) is going on, on \tilde{x} . ξ_1 is obtained by orthonormalization of the set $\{y_i', x_{i+1}', u_i'\}$, conditioned on the data $y(0, i-1)$, sequentially from right to left.

$$[y_i', x_{i+1}', u_i'] = \xi_1' C_1' + [\tilde{y}_i' | \tilde{x}_{i+1}' | \tilde{u}_i' | \tilde{u}_{i-1}'] \quad (7)$$

Similarly

$$\tilde{x} = C_2 \xi_2$$

is obtained by orthonormalizing from left to right

$$[y_i', x_{i+1}', u_i'] = \xi_2' C_2' + [\tilde{y}_i' | \tilde{x}_{i+1}' | \tilde{u}_i' | \tilde{u}_{i-1}'] \quad (8)$$

and the ξ variables are (see [1] for more details)

$$\xi_1' = (\tilde{y}_i', \tilde{x}_{i+1}', \tilde{u}_i')$$

$$\xi_2' = (\tilde{v}_i', \tilde{x}_{i+1}', \tilde{u}_i')$$

In essence $C_k C_k'$ is the covariance of the random variables $x(k = 1, 2)$. Now we could also work with the inverse relations:

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$$\underline{\xi}_k = C_k^{-1} \underline{\tilde{x}}_k (k = 1, 2)$$

leading to the so-called information filter. These two representations are dual; we can distinguish two types: inversion duality versus the standard duality [1]. Note that this approach leads to one-step predicted estimates $\hat{x}_{i+1|i}$ and errors (thus innovations). Alternatively we could work with filtered estimates [1]: $\hat{x}_i|i$, or with λ step predicted/smoothed estimate [4].

Suppose now that we do not know the actual covariances as given in (3) and (4); we use a model containing the estimated covariances:

$$P_0 \text{ and } \begin{bmatrix} Q_1 & C_1' \\ C_1 & R_1 \end{bmatrix} \quad (9)$$

Similarly, the system parameters may be incorrectly known, so that we propagate the Kalman filter (and thus the array's C_1 and C_2) as if the system model were:

$$x_{i+1} = \phi_1 x_i + \Gamma_1 u_i \quad \phi_1: n \times n \quad \Gamma_1: n \times p \quad (10)$$

$$y_i = H_1 x_i + v_i \quad H_1: m \times n \quad (11)$$

Therefore, x as computed by the filter is not the minimum variance estimate, and there will be a discrepancy between the computed and the actual error covariances. This deterioration of filter accuracy can ultimately lead to filter divergence. As a measure for this discrepancy we define the variance-ratios Σ_p

$$\Sigma_p = (P^a)^{-1/2} P^{1/2} \quad (P^a = \text{actual error covariance.}) \quad (12)$$

In our geometric picture, the variables $C_k^{-1} \underline{\tilde{x}}_k = \underline{n}_k$ for $k = 1, 2$, (we reserve $\underline{\xi}$ for truly orthonormal variables in this paper) will no longer be orthonormal.

$\tilde{x}_1 = x_1 - \hat{x}_1|i-1$ will still equal $P_1^{1/2} \chi_1$, where $P_1^{1/2}$ is the square root of the error covariance as calculated by the filter, but \tilde{x}_1 will not have unit covariance. The square root of $\text{cov } \tilde{X}$ will be the variance ratio

$$L_1 = P_1^{-1/2} (P_1^a)^{1/2}$$

We present algorithms in which L_1 is propagated as an entry in arrays which express the n -random variables in terms of truly orthonormal sets,

$$\underline{n}_k = M_k \underline{\xi}_k \quad k = 1, 2 \quad (13)$$

Obviously, just as with the filter itself, we can implement this propagation in either the covariance from given in (13) or in information form

$$\underline{\xi}_k = A_k \underline{n}_k \quad k = 1, 2 \quad (14)$$

in this case we propagate the $\Sigma_p = L^{-1}$. Either

filter implementation may be combined with either error analysis implementation, leading to four possibilities. Virtually, all square-root filter/error analysis algorithms proposed to date can be viewed as one of these four types. This is summarized in the following tableau:

Filter Implementation / Error Analysis Implementation	Covariance Form	Information Filter form
Covariance Type Form	Thornton (1976) (Propagates $x - \hat{x}$ rather than normalized \tilde{x} and uses UDU^T rather than $P^{1/2} P^{T/2}$), Section 3 -	Bierman (1974) Nishimura & Nead (1973) Curkendall (1972) Inversion Duality Morf-Kailath (1975)
Information Type Form	Section 2 of this paper	Inversion Duality [6]

The remainder of the paper is organized as follows: Section II examines in detail the filtering algorithm for the special case

$\{\phi_1, \Gamma_1, H_1\} = \{\phi_1^a, \Gamma_1^a, H_1^a\}$ and $C = C^a = 0$, as in the rest of the paper. The derivation is given in covariance-information form, and the covariance-covariance arrays are quoted (for C and $C^a \neq 0$, see [6]).

Section III extends these ideas to the general case of incorrect statistics, as well as model. Finally, Section IV mentions the fixed-interval smoothing problem.

II. ERROR ANALYSIS OF FILTERING WITH INCORRECT STATISTICS ONLY

The Pre-Array C_1

Defining

$$\tilde{X}' = [y_1' - \hat{y}_1' | [x_1, u_1, y[0, i-1]]] \cdot x_1' - \hat{x}_1' | [u_1, y[0, i-1]]' \\ u_1' - \hat{u}_1' | y[0, i-1]]'$$

then it is readily shown that

$$\tilde{X} = \begin{bmatrix} I & H_1 & 0 \\ 0 & \phi_1 & \Gamma_1 \\ 0 & 0 & I \end{bmatrix} \tilde{X}_1 \quad (15)$$

and

$$\text{cov } \tilde{X}_1 = \text{Block diag}\{R_1^a, P_1^a, Q_1^a\} \quad (16)$$

2

The "filter algorithm" rescales the \tilde{x}_1 -variables, setting

$$\tilde{x}_1 = \text{Block diag}(R_1^{1/2}, P_1^{1/2}, Q_1^{1/2}) \eta_1 = P^{1/2} \cdot \eta_1.$$

This leads to:

$$\tilde{x} = C_1 \eta_1 \quad (17)$$

(time index is suppressed), however, by virtue of (16), we see that $\text{cov } \eta_1 \neq I$ in general. Let $\xi_1 \xi_1'$ be a decomposition of $\text{cov } \tilde{x}_1$, then

$$\xi_1 = \xi^{-1} \tilde{x} = \xi^{-1} P^{1/2} \Delta \eta_1 = \Lambda_1 \eta_1 \quad (18)$$

is again an orthonormal set. Taking the upper triangular factor for ξ yields finally

$$[\tilde{x}', \xi_1'] = \eta_1' [C_1', \Lambda_1'] \quad (19)$$

where

$$\Lambda_1 = \text{Block diag}((R^a)^{-1/2} R^{1/2}, (P^a)^{-1/2} P^{1/2}, (Q^a)^{-1/2} Q^{1/2})$$

$$\Lambda_1 = \text{Block diag}(\Sigma_R, \Sigma_P, \Sigma_Q) \quad (20)$$

Remarks

1. In case $C^a \neq 0$, and/or $C \neq 0$, Λ_1 contains off-diagonal blocks. The 2-2 block entry remains Σ_P however.
2. Having obtained P_1 (the computed covariance) and Σ_P , we are able to find the actual covariance P_1^a .
3. Effect of the propagation algorithm:

$$\begin{aligned} [\tilde{x}', \xi_1'] &= [\eta_1' T] [T' C_1', T' \Lambda_1'] \\ &= [\eta_1' T] [C_2', T' \Lambda_1'] \end{aligned} \quad (21)$$

Thus, in the process of making C_1 lower triangular, we perform the same operations to Λ_1 .

The Post-Array C_2

Similarly, defining \tilde{y} by:

$$\tilde{y}' = [\epsilon_1', \tilde{x}_{i+1}' | i, \tilde{u}_1' | [y[0, i-1], \epsilon_1', \tilde{x}_{i+1}' | i]]$$

where ϵ_1 are the innovations, we can find:

$$\text{cov } \tilde{y} = \begin{bmatrix} R_\epsilon^a & R_\epsilon^a (\Delta K_g)' - R_\epsilon^a \Delta K_g' P_{i+1}^{-1} \Gamma Q \\ (\Delta K_g) \cdot R_\epsilon^a & P_{i+1} | i & P_{i+1}^a | i \cdot \Delta' \\ \Gamma Q' P_{i+1}^{-1} \Delta K_g R_\epsilon^a & \Delta \cdot P_{i+1}^a | i & \bar{Q}^a + \Delta P_{i+1}^a | i \Delta' \end{bmatrix} \quad (22)$$

where

$$\begin{aligned} \Delta K_g &= \phi_1 (P_{i+1}^a | i-1 H_1' - P_{i+1} | i-1 H_1' R_{i+1}^{-\epsilon} R_\epsilon^a) \\ R_{\epsilon_1}^a &= H_1 P_{i+1}^a | i-1 H_1' + R_{i+1}^a \end{aligned} \quad (23)$$

$$\bar{Q}_1^a = Q_1^a - Q_1^a \Gamma_1' P_{i+1}^{-a} \Gamma_1 Q_1^a$$

$$\Delta = Q_1^a \Gamma_1' P_{i+1}^{-a} - Q_1 \Gamma_1' P_{i+1}^{-1}$$

The Δ -terms in (22) vanish when correct statistics are used. Rescaling according to the assumed covariances yields

$$\tilde{y}_1 = \text{Block diag}(R_1^{\epsilon/2}, P_{i+1}^{1/2} | i, Q_1^{1/2}) \eta_2 \quad (24)$$

and finally

$$\tilde{x} = C_2 \eta_2 \quad (25)$$

Now again $\text{cov } \eta_2 \neq I$ in general, and reorthonormalization is needed to yield the orthonormal set

$$\xi_2 \stackrel{\Delta}{=} \Lambda_2 \eta_2 \quad (26)$$

thus we have

$$[\tilde{x}', \xi_2'] = \eta_2' [C_2', \Lambda_2'] \quad (27)$$

From Remark 3 of the previous paragraph:

$$\eta_1' T = \eta_2'$$

η_1 and η_2 span the same space, and ξ_k , $k=1,2$ are both orthonormal bases for this space. They are therefore related by an orthogonal transformation \bar{T} . Thus

$$\bar{T} \xi_1 = \xi_2$$

and

$$\bar{T} \Lambda_1 T = \Lambda_2$$

Since we are basically interested in Σ_P , we can consider factoring (22) in a (permuted) triangular form SS' with

$$S = \begin{bmatrix} \Delta_{11} & X & X \\ 0 & \Delta_{22} & 0 \\ 0 & X & \Delta_{33} \end{bmatrix}$$

then the (2,2)-element of Λ_2 is exactly $(\Sigma_P)_{i+1}$ and can be obtained from S similarly to (18). Therefore, we do not need to perform the complete transformation \bar{T} on the $(m+n+p)$ -size matrix $(\Lambda_1 T)$, but a reduced \bar{T}^r of size n suffices to yield $(\Sigma_P)_{i+1}$.

Remark.

1. From the identity $C_1 \text{cov } \eta_1 C_1' = C_2$ $\text{cov } \eta_2 C_2' = \text{cov } \tilde{x}$ we obtain the identities

$$R_{\epsilon_1}^a = R_{i+1}^a + H_{i+1} P_{i+1}^a H_{i+1}'$$

$$\begin{aligned} P_{i+1}^a &= (\phi_1 - K_{g_1} H_{i+1}) P_{i+1}^a (\phi_1 - K_{g_1} H_{i+1})' + \Gamma_{i+1} Q_{i+1}^a \Gamma_{i+1}' \\ &\quad + K_{g_1} R_{i+1}^a K_{g_1}' \end{aligned}$$

3

We could have started from these equations to get the array's without going to the geometrical description.

2. For the covariance-covariance form we simply invert (18) and (26) to get:

$$\tilde{x} = C_2 \eta_k, \quad \eta_k = \Lambda_k^{-1} \tilde{\xi}_k, \quad k = 1, 2.$$

III. GENERAL MODELING ERRORS

We now consider errors in our assumption in the statistics and the model parameters. For simplicity however, we assume here that $C = C^a = 0$; non-zero case, see [6]. In order to reduce the notational burden we omit the explicit time dependence of the model parameters. As a first step in obtaining a representation of η_1 in terms of an orthonormal set, we invert the equation $\tilde{x} = C_1 \eta_1$ to obtain

$$\tilde{y}_1 = R^{-1/2} [(\Delta H - H\Phi^{-1}\Delta\Phi)x_1 - H\Phi^{-1}\Delta\Gamma u_1 + v_1]$$

$$\tilde{x}_1^+ = P_1^{-1/2} [(x_1 - \hat{x}_1 | i-1) + \Phi^{-1}\Delta\Phi x_1 + \Phi^{-1}\Delta\Gamma u_1]$$

$$u_1 | i-1 = \tilde{x}_1 + P_1^{-1/2} \Phi^{-1}\Delta\Phi x_1 + P_1^{-1/2} \Phi^{-1}\Delta\Gamma u_1$$

Here

$$\Delta\Phi = \Phi^a - \Phi; \quad \Delta H = H^a - H; \quad \Delta\Gamma = \Gamma^a - \Gamma.$$

A new subtle point arises, when the system matrices are mismodeled, \tilde{x}_1^+ , the pre-array variable above, is no longer equal to \tilde{x}_1 , the post-array variable of the previous time step.

Furthermore, we see that to obtain a self-contained recursion, we must augment the η_1 variables with the actual state x_1 .

Suppose we have available a representation for the augmented state in terms of orthonormal variables ξ_k

$$\begin{bmatrix} \tilde{x}_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} L_\Delta & 0 \\ L_{x\Delta} & L_x \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \quad (28)$$

$$u_1 = (Q^a)^{1/2} \xi_3, \quad v_1 = (R)^{1/2} \xi_4$$

are certainly orthogonal to the augmented state. Therefore using (28) and the equation

$$x_{i+1} = \Phi^a x_i + \Gamma^a u_i$$

we can set up the following representation

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{x}_1^+ \\ \tilde{u}_1 \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} R^{-1/2}(\Delta H - H\Phi^{-1}\Delta\Phi)L_{x\Delta} & R^{-1/2}(\Delta H - H\Phi^{-1}\Delta\Phi)L_x & & \\ L_\Delta + P_1^{-1/2}\Phi^{-1}\Delta\Phi L_{x\Delta} & P_1^{-1/2}\Phi^{-1}\Delta\Phi L_x & & \\ 0 & 0 & & \\ \Phi^a L_{x\Delta} & \Phi^a L_x & & \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}$$

$$\begin{bmatrix} -R^{-1/2}H\Phi^{-1}\Delta\Gamma(Q^a)^{1/2} & R^{-1/2}(R^a)^{1/2} & & \\ P_1^{-1/2}\Phi^{-1}\Delta\Gamma(Q^a)^{1/2} & 0 & & \\ Q^{-1/2}(Q^a)^{1/2} & 0 & & \\ \Gamma^a(Q^a)^{1/2} & 0 & & \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} \quad (29)$$

This corresponds (in augmented form) to Eq. (18) of Section II. The above array corresponds to $(\Lambda_1^*)^{-1}$.

We apply on the left of the above array the orthogonal transformation which updates η_1 to η_2 ; that is, we apply the transpose of the transformation which updates the filter array C_1 to C_2 , thus

$$\begin{bmatrix} v_1 \\ \tilde{x}_{i+1} \\ \tilde{u}_1 \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} X & X & X & X \\ S_{21} & S_{22} & S_{23} & S_{24} \\ X & X & X & X \\ \Phi^a L_{x\Delta} & \Phi^a L_x & \Gamma^a(Q^a)^{1/2} & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} \quad (30)$$

(The X's denote entries of no particular interest.) We now apply orthogonal transformation on the right of the above array to bring the second and fourth block rows to the form

$$\begin{bmatrix} \tilde{x}_{i+1} \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} L_\Delta & 0 & 0 & 0 \\ L_{x\Delta} & L_x & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}$$

This yields L_Δ , the square-root covariance of \tilde{x}_{i+1} and updates (28) to time $i+1$. We can easily initialize (28) at time 0, when $\tilde{x}_0 = x_0 = P_0^a \xi$.

Remark.

1. There is some redundancy in considering $\Delta\Gamma \neq 0$ and $Q^a \neq Q$ together, because Γ and Q only arrive in the combination $\Gamma Q^{1/2}$. So, we can always define a $Q_{\text{equiv}} = I$, $\Gamma_{\text{equiv}} = \Gamma Q^{1/2}$. In this case, $\Delta\Gamma \neq 0$, but $Q_{\text{equiv}} = Q_{\text{equiv}}^a$. Alternatively, if $(\Phi, \Gamma Q^{1/2})$ forms a controllable pair, then we can set $Q_{\text{equiv}} = \Gamma Q^{1/2}$ and $\Gamma_{\text{equiv}} = I$ thus $\Delta\Gamma = 0$ if both $(\Phi^a, \Gamma^a Q^{a1/2})$ are controllable.

2. Choosing $\Phi_i x_i$ as the "augmenting" state to \tilde{x} , and expressing $x^* = [(\Phi_i x_i)', \tilde{x}']$ in terms of $\eta^* = [\hat{x}_1 | i-1, \eta_1']$ gives an augmented pre-array C_1^* which contains C_1 , and is itself in upper triangular form (note that $\hat{x}_1 | i-1 \perp \tilde{x}_1 | i-1$ with correct model and statistics). Similarly, we express $x^* = [x', (\Phi_i x_i)']$ in terms of $\eta_2^* = [\eta_2', x_{i+1} | i]$ which yields a lower triangular C_2^* with C_2 contained in it

$$\tilde{x}^* = \begin{bmatrix} \phi \Sigma^{1/2} & 0 & \phi P^{1/2} & 0 \\ 0 & R^{1/2} & HP^{1/2} & 0 \\ 0 & 0 & \phi P^{1/2} & \Gamma Q^{1/2} \\ 0 & 0 & 0 & Q^{1/2} \end{bmatrix} \tilde{\eta}_1^*$$

$$\tilde{x}^* = \begin{bmatrix} R^{c/2} & 0 & 0 & 0 \\ \tilde{K} & P_{i+1}^{1/2} & 0 & 0 \\ D & E & Q^{-1/2} & 0 \\ -\Gamma D & P_{i+1}^{1/2} & -\Gamma E & -\Gamma Q^{-1/2} & I \end{bmatrix}$$

The augmented Λ -arrays are quite involved however, see [6].

IV. SMOOTHING

We now briefly describe how to extend the algorithm of Section III to the square-root form of the two-filter fixed-interval smoother. Each filter will have an error analysis addendum propagated according to the idea of Section III. Here we will just point out the modifications needed in the separate filters to facilitate a smoothing solution and demonstrate how to combine information from the two filters to obtain an Λ_Δ for the smoothed estimate.

The backward filter is implemented in information form since it is initialized with infinite a priori covariance. However, the η_k $k = 1, 2$ variables are the same for both information and covariance filters. Therefore, when the only modeling error is in the statistics, the error analysis algorithm is unchanged from Section II. When there are mismodeled system matrices and the η are augmented by x_1 , it must be remembered that u_1 is not orthogonal to x_{i+1} . Probably the most straightforward way to obtain a representation for $(\eta_1^*, x^*)'$ in terms of an orthonormal set (that is to obtain the array analogous to the inverse of (18), (26) is to decompose u_1 as

$$u_1 = Q_{11}^{a,a'} \Pi_{i+1}^{-1} x_{i+1} + \tilde{u}_1 | x_{i+1}, \quad \Pi_{i+1} = \text{cov}(x_{i+1}) \quad (31)$$

where $\tilde{u}_1 | x_{i+1}$ is orthogonal to u_{i+1} and has covariance

$$Q_1^b = (Q_1^a - Q_1^a \Gamma_1^a \Pi_{i+1}^{-1} \Gamma_1^a Q_1^a)$$

When there are mismodeled system matrices, the augmented state in both forward and backward error analysis arrays should be represented in the form

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \Pi^{1/2} & 0 \\ M_{\Delta x} & M_\Delta \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad \text{rather than} \quad \begin{bmatrix} \tilde{x} \\ x \end{bmatrix} = \begin{bmatrix} L_\Delta & 0 \\ L_{x\Delta} & L_x \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad (32)$$

Decomposing each \tilde{x}_1 into a component correlated with and a component orthogonal to the actual state enables us to get a joint representation for \tilde{x}^f (from forward filter) and \tilde{x}^b (from backward

filter). We have the separate representations

$$\begin{bmatrix} x_1 \\ \tilde{x}_1^f \end{bmatrix} = \begin{bmatrix} \Pi_1^{1/2} & 0 \\ M_{\Delta x}^f & M_\Delta^f \end{bmatrix} \begin{bmatrix} \xi_1^f \\ \xi_2^f \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ \tilde{x}_1^b \end{bmatrix} = \begin{bmatrix} \Pi_1^{1/2} & 0 \\ M_{\Delta x}^b & M_\Delta^b \end{bmatrix} \begin{bmatrix} \xi_1^b \\ \xi_2^b \end{bmatrix} \quad (33)$$

ξ_1^f must be related to ξ_1^b by an orthogonal transformation. $\xi_1^f = \xi_1^b$ if we insist on a unique form for Π_1^f , say lower triangular with positive diagonal elements. Some study reveals [6] that ξ_1^f is a function of $\{u_0, \dots, u_{i-1}\}$ and $\{v_0, \dots, v_{i-1}\}$, while ξ_1^b is a function of $\{u_1, \dots, u_N\}$ and $\{v_1, \dots, v_N\}$ where the smoothing interval is $i = 1, \dots, N$. Thus ξ_2^f and ξ_2^b are orthogonal and we may write

$$\begin{bmatrix} \tilde{x}_1^b \\ \tilde{x}_1^f \end{bmatrix} = \begin{bmatrix} M_{\Delta x}^b & M_\Delta^b & 0 \\ M_{\Delta x}^f & 0 & M_\Delta^f \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2^b \\ \xi_2^f \end{bmatrix} \quad (34)$$

When we have only incorrect modeling of statistics this simplifies to

$$\begin{bmatrix} \tilde{x}_1^b \\ \tilde{x}_1^f \end{bmatrix} = \begin{bmatrix} L^b & 0 \\ 0 & L_\Delta^f \end{bmatrix} \begin{bmatrix} \xi^b \\ \xi^f \end{bmatrix}$$

The forward filter may be implemented in either covariance or information form. For purposes of illustration, let us assume a covariance forward filter and an information backward filter. To combine the estimates of the two filters to obtain a smoothed estimate we may interpret

$(P_1^b)^{-1/2} \tilde{x}_1^b | i$ as an additional measurement with output matrix $(P_1^b)^{-1/2}$ which is to be used to update $\hat{x}_{i+1|i-1}^f$ [5]. This interpretation is illustrated below with corresponding quantities one under the other

$$\begin{aligned} y_1 &= H x_1 + v_1 \\ d_1^b &= (P_1^b)^{-1/2} x_1 + \tilde{x}_1^b \end{aligned} \quad (35)$$

The "measurement noise" \tilde{x}_1^b is assumed by the filter to have unit covariance; so, the appropriate measurement update is given by the following arrays:

$$\begin{bmatrix} d_1^b - (P_1^b)^{-1/2} \hat{x}_{i+1|i-1}^f \\ x_1 - \hat{x}_{i+1|i-1}^f \end{bmatrix} = \begin{bmatrix} I & (P_1^b)^{-1/2} (P_1^f)^{1/2} \\ 0 & (P_1^f)^{1/2} \end{bmatrix} \begin{bmatrix} \tilde{x}_1^b \\ \tilde{x}_1^f \end{bmatrix} \quad (36)$$

Lower triangularizing (36) with orthogonal transformations on the right gives the post-array

$$\begin{bmatrix} d_1^b - (P_1^b)^{-1/2} \hat{x}_{i+1|i-1}^f \\ x_1 - \hat{x}_{i+1|i-1}^f \end{bmatrix} = \begin{bmatrix} v_1 & 0 \\ K^* & (P_1^s)^{1/2} \end{bmatrix} \begin{bmatrix} v_1^s \\ \tilde{x}_1^s \end{bmatrix}$$

The smoothed estimate of x_1 is

$$\hat{x}_1^s = \hat{x}_{1|1-1}^f + K_{11}^{*-1} [d_1^b - (P_1^b)^{-1/2} \hat{x}_{1|1-1}^f] \quad (37)$$

and

$$x_1 - \hat{x}_1^s = (P_1^s)^{1/2} \tilde{x}_1^s.$$

The set $\{\tilde{x}_1^s, v_1^s\}$ is obtained from the set $\{\tilde{x}_1^b, \tilde{x}_1^f\}$ through the array update transformation. Applying the same transformation to the representation (34) for $\{\tilde{x}_1^b, \tilde{x}_1^f\}$ yields

$$\begin{bmatrix} v_1^s \\ \tilde{x}_1^s \end{bmatrix} = \begin{bmatrix} X & X & X \\ S_{21} & S_{22} & S_{23} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2^b \\ \xi_2^f \end{bmatrix}$$

An additional orthogonal transformation applied on the right will give

$$\tilde{x}_1^s = \begin{bmatrix} L_\Delta^s & 0 & 0 \\ & \bar{\xi}_2 \\ & \bar{\xi}_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2^b \\ \xi_2^f \end{bmatrix} \quad (38)$$

$$(P_s^a)^{1/2} = (P_1^s)^{1/2} L_\Delta^s.$$

V. CONCLUSION

By viewing the square-root arrays as representations of the estimate residuals in terms of orthonormal sets of random variables we have expressed a framework for the known square-root error analysis algorithm. This framework led us to a new algorithm presented herein which is compatible with either the covariance or information implementations of the filter.

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